OKLAHOMA MATH DAY

NOVEMBER 22, 2003

SOONER MATH BOWL — SOLUTIONS

Stage 1, Round 1. Question 1. How many triangles?

Solution. Triangles of size 1: $1 + 3 + 5 + 7 = 16$

Triangles of size 2: 7 (one of 'em is upside down)

Triangles of size 3: 3

Triangles of size 4: 1

Total = 27. □

Question 2. Magic square summing to 21.

Solution. Just add 2 to every entry in the given magic square.

□

Stage 1, Round 2 (Blitz).

(a) In general, if today is Saturday, then the same date next year is Sunday (since $365 = 52 \cdot 7 + 1$). But 2004 is a leap year, so November 22nd, 2005 is Tuesday.

(b) $\frac{6}{7} < 1$ so any positive power is also less than 1. $\frac{7}{6} > 1$, so any positive power is bigger than 1.

(c) Draw a line outwards from A. Any such line that is not tangent to the curve intersects it an even number of times. This means that the point is **outside** the curve.

(d) Maximum number of socks is 3. Any more would guarantee a pair of some color.

(e) If a cubic has roots α, β and γ, then we have the equation

$$(x-\alpha)(x-\beta)(x-\gamma) = x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\alpha\gamma)x - \alpha\beta\gamma$$

So we see by comparison that the coefficient of x^2 is multiplied by 2, the coefficient of x is multiplied by $2^2 = 4$ and the constant term is multiplied by 8. Therefore, the required cubic is

$$x^3 - 10x^2 + 20x - 8 = 0$$

(f) If $\sin(\theta) = \frac{1}{5}$, then $\cos(\theta) = \sqrt{1-\sin^2(\theta)} = \frac{\sqrt{24}}{5}$. Therefore, $\tan(\theta) = \frac{1}{\sqrt{24}}$.

(g) The arc-length is 2, while the circumference is $2 \cdot 2\pi = 4\pi$. Thus, this sector is $\frac{2}{4\pi}$-fraction of the full circle with radius 2. So its area is $\frac{2}{4\pi}$ times the area of the circle which is $\frac{2}{4\pi} \cdot \pi \cdot 2^2 = 2$.

Stage 1, Round 3.

Solution to Question 1.

Let V be the volume of the first glass, so the amount of wine in it is $\frac{V}{2}$. The volume of the second glass is $2V$ and the amount of wine in it is $\frac{1}{4} \cdot 2V = \frac{V}{2}$, also. The total amount of wine in the third container is $\frac{V}{2} + \frac{V}{2} = V$ out of a total volume of $V + 2V = 3V$, so the fraction of wine in the mixture is $\frac{1}{3}$ or 33.33%.

Solution to Question 2.

By comparing the face with two dots in the second picture with the first and third pictures of the die, it follows that there must be at least two faces each with two dots on them. But we also see faces with 6, 1, 3 and 4 dots, so we have accounted for all six faces.

Therefore the sum is $1 + 2 + 2 + 3 + 4 + 6 = 18$.

Stage 2.

Solution to Question 1.

By similarity of smallest and largest right triangles in the figure, we obtain that

$$\frac{x}{2} = \frac{2}{x+3}$$

which yields the quadratic equation $x^2+3x-4=0$ or $(x+4)(x-1)=0$, which yields the positive solution $x=1$.

Solution to Question 2.

A light bulb will be off if its switch has been toggled an even number of times (since all light bulbs are unlit to start out with). The number of times the switch numbered k gets toggled is precisely equal to the number of factors of k.

The only numbers that have an odd number of factors are numbers that are perfect squares. Therefore, the only lights that will remain on are the ones numbered 1, 4 and 9.

Solution to Question 3.

The number of zeros at the end of a number is simply the highest power of 10 that divides evenly into it. Powers of 10 arise as multiples of 5 times 2 or multiples of 25 times 4 or multiples of 125 times 8 etc. In our case we only care about multiples of 5 and 25. All multiples of 5, except 25 and 50, contribute a single zero, while 25 and 50 contribute 2 zeros. This gives a total of 12 zeros.

Stage 3.

$$4^{4-4} + 4 = 5$$

Stage 3.

$$9^{9^9}$$

Stage 3.

$\frac{\sqrt{11}-\sqrt{13}}{2}$ is between -1 and 0, so the next integer is 0.

Stage 3.

Since Volume 1 is shelved to the left of Volume 2, the bookworm has only to travel through the front cover of Volume 1 and the back cover of Volume 2, which is a total of $\frac{2}{8} = \frac{1}{4}$ inch. So total time to eat through is 1 day.

Stage 3.

It is not too difficult to see algebraically that the digits of such a number must add to 11. The following are all such pairs that add to the same perfect square, 121.

$$(47, 74), (38, 83), (56, 65)$$

Stage 3.

By symmetry the maximum height is obtained by pulling the string at the middle.

Pulling up a string creates two right triangles and we want to estimate the height of either right triangle. The base is half the length of the court which is 47 feet. The hypotenuse is 1 inch longer than the length of half the court. By Pythagoras' theorem, the height (in inches) is $\sqrt{(47 \cdot 12 + 1)^2 - (47 \cdot 12)^2} = \sqrt{2 \cdot 47 \cdot 12 + 1}$ which is approximately 33 inches or nearly 3 feet.

Stage 3.

The radius of the circle is $\frac{\sqrt{2}}{2}$ so its area is $\frac{\pi}{2}$.

Stage 3.

Think of the original cube as being composed of 27 unit cubes. We have removed a total of 7 of these (one from each face and the central cube). Of the remaining cubes the 8 vertex cubes each contribute 3 faces and the other 12 cubes each contribute 4 faces. This gives a total of $8 \cdot 3 + 12 \cdot 4 = 72$ square units.

Stage 3.

The radius of the circle is $\frac{1}{2}$ so its area is $\frac{\pi}{4}$.