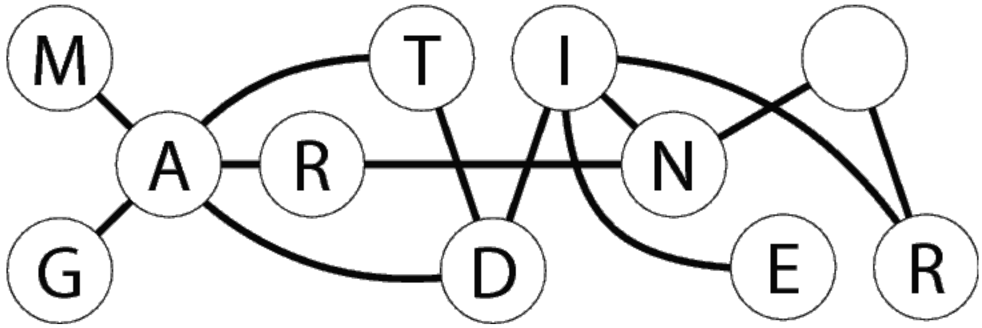


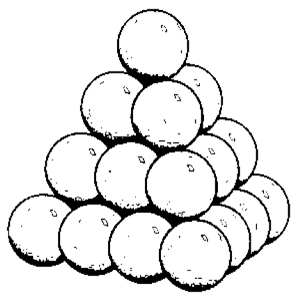
Photo Martin Gardner by Alex Bellos in 2008 in Norman

Born in Tulsa in 1914 and passed away in Norman in 2010.

Stage 1

Stage 1, Round 1 (2 Questions, 3 Minutes)

1. After a long day at the office, President Boren likes to relax by stacking golden spheres in a pyramid shape. For example, if there are 4 spheres on one side of the base, then when he's done it looks like:



The more stressed President Boren is, the more spheres he stacks!

a. When the Oklahoma football team lost to Kansas State, President Boren stacked a pyramid with 6 spheres on each side of the base. How many spheres did President Boren use?

b. When the Oklahoma football team lost to Notre Dame, he stacked spheres into a pyramid which was 10 spheres *tall*. How many spheres did President Boren use?

c. In general, if the President stacks spheres into a pyramid with n spheres on one side of the base, then please give a formula which calculates the number of golden spheres in President Boren's pyramid.

The Answer: If there are n balls on one side of the base, then there are

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

In particular, when $n = 6$ we have 91 and when $n = 10$ we have 385.

2. For this question we will assume that when a baby is born they are equally likely to be a boy or a girl. Mike has two children.

(a) What is the probability at least one of them is a girl?

(b) If I tell you the oldest one is a girl, what is the probability both are girls?

(c) If I tell you one of them is a girl, what is the probability both are girls?

The Answer: The probabilities are 3/4, 1/2, and 1/3, respectively.

Stage 1, Round 2 (Blitz Round, 3 Minutes)

a. If you roll two dice and add the numbers you get, what is the probability you will get exactly 6?

The Answer: The probability is 5/36.

b. If you cut a circular cake with three straight cuts (all distinct, all nontrivial), what is the *minimum* number of pieces you can have at the end?

The Answer: 4 pieces.

c. A $3 \times 3 \times 3$ cube has all its sides painted red. If it is cut into $1 \times 1 \times 1$ cubes, how many of those little cubes will be red on *at least* two sides?

The Answer: 20 cubes.

d. Let $x = 2012^5$. Which of the following numbers is closest to x:

- 2,000
- 20,000,000,000
- 2,000,000,000,000
- 2,000,000,000,000,000

The Answer: By doing orders of magnitude, we see $2,000,000,000,000,000$ is closest.

e. Your boss offers to give you a raise. You can either get a single 25% raise or get two raises, one immediately after the other. The two raises would be a 10% raise and a 15% raise. Is it better to get the 25% raise, or the two raises, or does it not matter?

The Answer: If you do two raises, then the second one compounds the first one. Doing it as two raises is the same as getting one raise of 26.5%.

f. If the polynomial $p(x) = x^{2012} + x^{15} + x^{11} + c$ has $x + 1$ as a factor, what is c?

The Answer: If $x + 1$ is a factor, then $x = -1$ is a root. Plugging in we get $c = 1$.

g. If you multiply together the prime numbers between 1 and 10, what do you get?

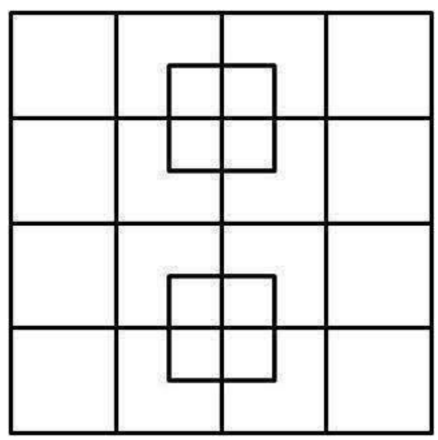
The Answer: The primes are $2, 3, 5, 7$, so their product is 210.

Stage 1, Round 3 (3 Questions, 5 Minutes)

1. Alex can saw a cylindrical log into 10 pieces in 10 minutes. How many pieces can he saw it into in 20 minutes?

   The Answer: If he cuts it into 10 pieces in 10 minutes, then this means he makes 9 cuts in 10 minutes. So he will make 18 cuts in 20 minutes and, hence, have 19 pieces.

2. Consider the following picture:



   (a) How many squares are in this picture?

   The Answer: We count them by size, using the sizing from part (b). There are 8 1×1 squares, 18 2×2 squares, 9 4×4 squares, 4 6×6 squares, and 1 8×8 square. The total is 40 squares.

   (b) Let's say the smallest square in the picture is 1 inch by 1 inch. Let $\mathcal{S}$ be the sum of the areas of all the squares of area less than or equal to 16 square inches and $\mathcal{B}$ is the sum of the areas of all the squares of area strictly greater than 16 square inches. Which is bigger, $\mathcal{S}$ or $\mathcal{B}$?

   The Answer: Adding up the areas, we see that $\mathcal{S} = 224$ square inches while $\mathcal{B} = 208$ square inches.

3. Remember that we write $i = \sqrt{-1}$. Please calculate z if

$$z = (i+1)^8 + (i-1)^8.$$

   The Answer: $z = 32$.

Lunch!

Stage 2

Stage 2, Round 1 (Blitz Round, 3 Minutes)

a. If you draw six dots and connect each pair of dots by a line segment, how many line segments have you drawn?

The Answer: You draw $\binom{6}{2} = \frac{6!}{2!4!} = 15$ line segments.

b. When you pick a 4 digit PIN number for your bank you can use any of the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. How many total possible numbers are there?

(Amazingly, one out of ten people pick 1234.)

The Answer: $10^4 = 10,000$ numbers.

c. Which is larger: 15^{11} or 11^{15}?

The Answer: By orders of magnitude, 11^{15} is larger.

d. Consider the sequence $a_1 = 3, a_2 = 7, a_3 = 15, a_4 = 31, a_5 = 63, \ldots$. If you continue this sequence, what is a_9?

The Answer: The rule is $a_k = 2a_{k-1} + 1$, so computing we get $a_9 = 1023$.

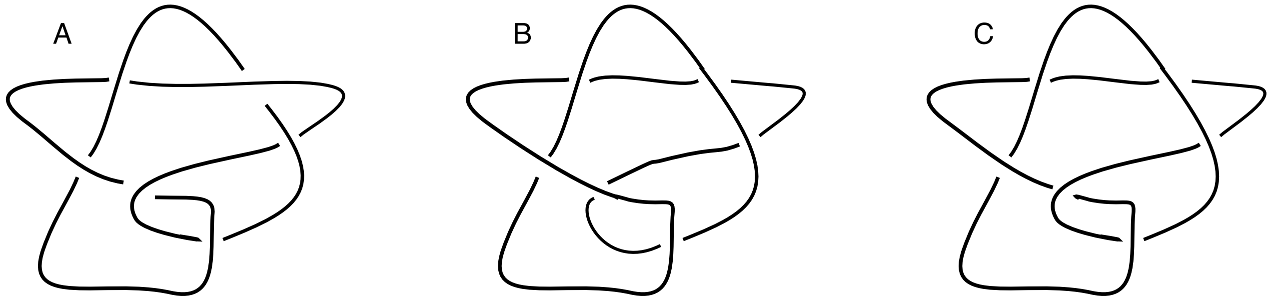
e. If circle A has area twice as big as circle B, how much bigger is the radius of circle A compared to the radius of circle B?

The Answer: The radius of the big circle is $\sqrt{2} \sim 1.414$ times the radius of the small circle.

f. Your boss offers to give you a raise. Because of strange budget rules, it has to be given in two parts. You get a 10% raise and a 15% raise. You get both raises at the same time, but one has to be entered into the computer first. Is it better to get the 10% raise first, then the 15% raise. Or vice versa? Or does it not matter?

The Answer: The order doesn't matter. If S is your current salary, then your new salary in the first case is $S(1.1)(1.15)$ and in the second case it is $S(1.15)(1.1)$.

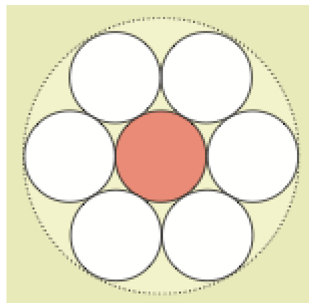
g. Which of the following knots can be untangled (without cutting) to a form a circle?



The Answer: Knot B.

Stage 2, Round 2 (3 Questions, 5 Minutes)

1. Seven circles of radius 1 are packed together inside a larger circle (see the picture below). What is the area of the region inside the large circle but outside the small circles?



The Answer: The radius of the big circle is 3, so the area of the region is $9\pi - 7\pi = 2\pi$.

2. The Euler function $\varphi(n)$ is defined to be the number of integers between 1 and n (including 1 and n) which do not share any common factor with n. For example, $\varphi(12) = 4$, since $1, 5, 7$, and 11 is the complete list of the numbers from among $1, 2, 3, \ldots, 12$ without a common factor with 12. What is $\varphi(100)$?

   The Answer: When you work it out, $\varphi(100) = 40$.

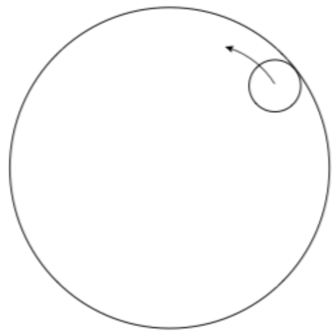
3. Jing is running for office in this year's election and distributed leaflets at each campaign stop. At each stop she distributed exactly half of all the leaflets she had left. If at the 7th event she gave out the last of her leaflets, how many leaflets did she start with?

   The Answer: Working backwards from the last stop, she gave away $1, 2, 4, 8, 16, 32$, and 64 leaflets. Adding these up, she started with 127 leaflets.

Stage 3

Stage 3, Round 1 (3 Questions, 5 Minutes)

1. In the picture below the smaller circle has a radius of 3 inches. If the small circle rolls around the inside of the big circle exactly once, it makes exactly 2012 revolutions. What is the area of the large circle?



The Answer: The circumference of the small circle is 6π, so it traveled a total of $2012 * 6\pi$ units. Hence the circumference of the big circle is $2012 * 6\pi$ and, hence, has radius $2012 * 3 = 6036$.

2. A bag contains 20 red marbles, 5 green marbles, and 8 black marbles.

   (a) If you randomly draw marbles from the bag with your eyes closed, how many do you have to draw to be certain that you have drawn at least one red marble?

   The Answer: In the worst case you first draw all the non-red marbles, so you need to draw 14 marbles to guarantee at least one red marble.

   (b) If you randomly draw marbles from the bag with your eyes closed, how many do you have to draw to be certain that you have drawn at least one marble of each color?

   The Answer: In the worst case you draw all the red marbles and all the black marbles before drawing a green marble, so you need to draw 29 marbles to guarantee at least one marble of each color.

3. It can be shown that there exists unique positive integers $m, n > 1$ such that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = m^2$$

Find n.

The Answer: Working it out, we find that $n = 24$ and $m = 70$

Stage 3, Round 2 (3 Questions, 5 Minutes)

1. Let $f(x) = x^2 - 1$. Let $F(x) = f(f(x))$. Find the zeros of $F(x)$.

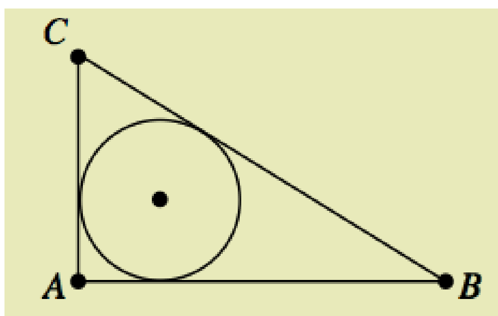
   The Answer: $f(f(x)) = (x^2 - 1)^2 - 1 = x^4 - 2x^2$ which has $x = 0, \pm\sqrt{2}$ as roots.

2. Please compute x:

$$x = 6 - \sqrt{6 - \sqrt{6 - \sqrt{6 - \sqrt{6 - \cdots}}}}.$$

   The Answer: Solving the corresponding quadratic yields $x = 4$ and $x = 9$, but clearly x is less than 6 so it must be that $x = 4$.

3. The triangle below is a right triangle. The side AB has length 4 and the side AC has length 3. What is the radius of the circle?



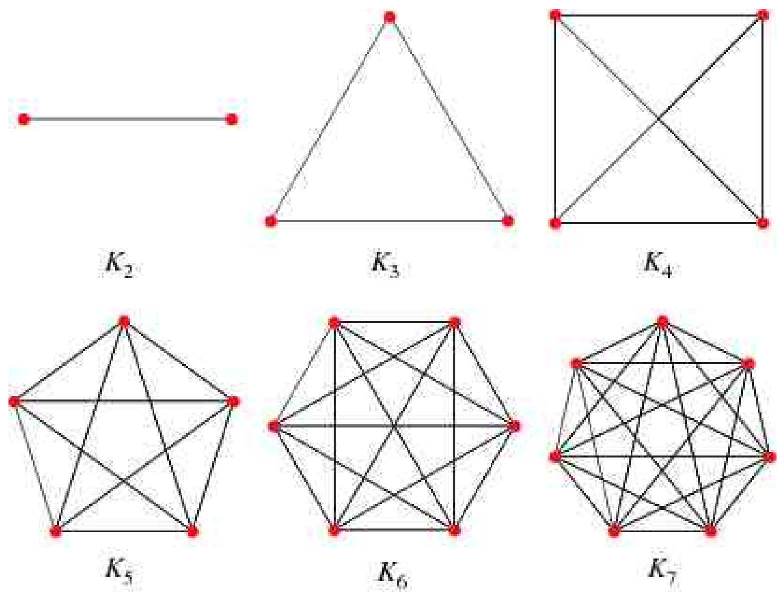
   The Answer: Let r be the radius. If you draw lines from the center of the circle to where the circle touches each side of the triangle, you can use geometry (*not trigonometry*!) to see that the hypotenuse of the triangle is $(3 - r) + (4 - r)$. But it's also equal to 5. Setting these equal and solving yields that $r = 1$.

The End!

Spot Prize I

Name:___________________ School:___________________

The *complete graph* on n vertices, K_n, is what you get by taking n points and connecting each of them with a straight line segment. The complete graphs K_2, K_3, K_4, K_5, K_6, and K_7 are shown below.



1. How many line segments are there in K_8, K_9, and K_{10}?

2. Please given a formula for the number of line segments in K_n for any n. Note that your formula should depend on n. If nobody gives a correct formula, then the person who has computed the most K_n's correctly will be judged the winner.

The Answer: There are

$$\binom{n}{2} = \frac{n!}{2!(n-2)!}$$

edges in K_n. In particular, $K_8 = 28$, $K_9 = 36$, and $K_{10} = 45$.

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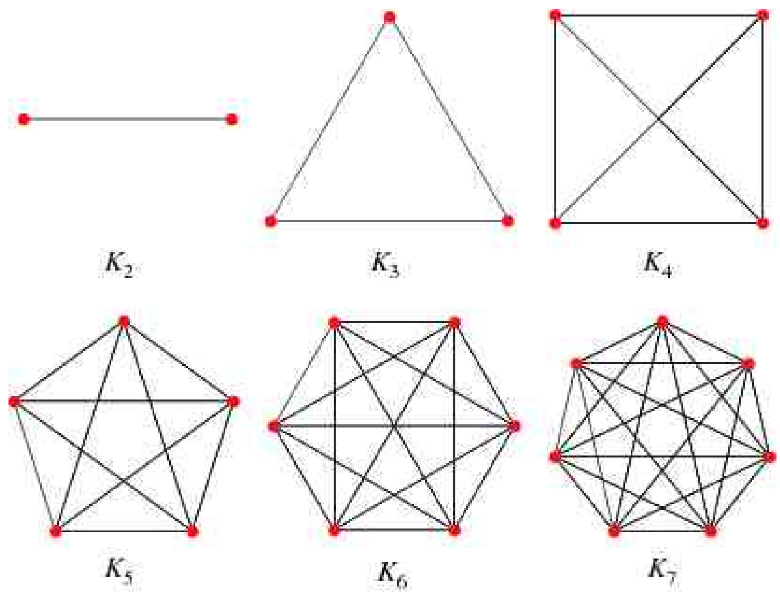

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Spot Prize II

Name:__________________ **School:__________________**

A famous question answered by Euler in 1735 is the Königsberg Bridges Problem. Is it possible to walk around the city on a route which crosses each bridge *exactly* once? Euler proved that in Königsberg the answer is No. However, in New York City the answer is Yes!

Here is a map of the NYC area with all the bridges drawn. Find a route which starts in Manhattan, ends in Brooklyn, and crosses each bridge exactly once! If you can find a route which starts in Manhattan, ends in Staten Island, and crosses only each bridge only once, we'll **double** your prize!

The Answer: There are various different paths you can draw from Manhattan to Brooklyn which only crosses each bridge once. However there are *no* paths which go from Manhattan and end in Staten Island. The reason is that every time you enter and leave one of the regions on the map, you cross *two* bridges (one entering, one leaving). So if a region has an odd number of bridges, a successful journey has to either begin or end at that region. In particular, for NYC this applies to both Manhattan and Brooklyn. So a successful journey has to begin at one and end at the other! If you google and look at a map of Königsberg you will see that there are more than two regions with an odd number of bridges. This makes a successful journey impossible!

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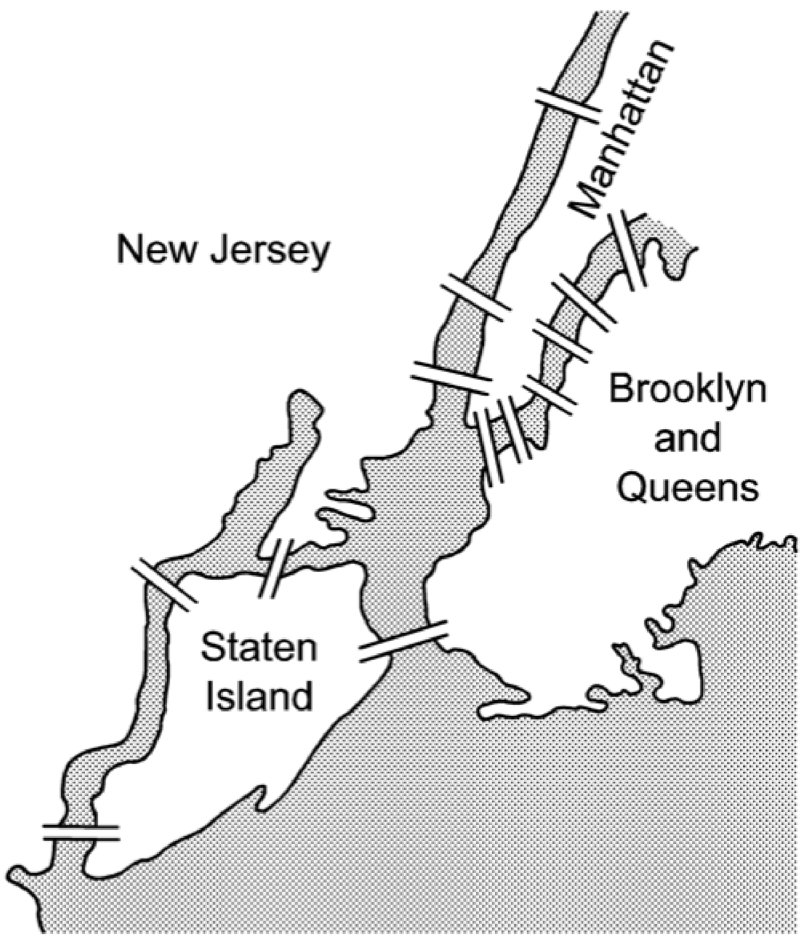

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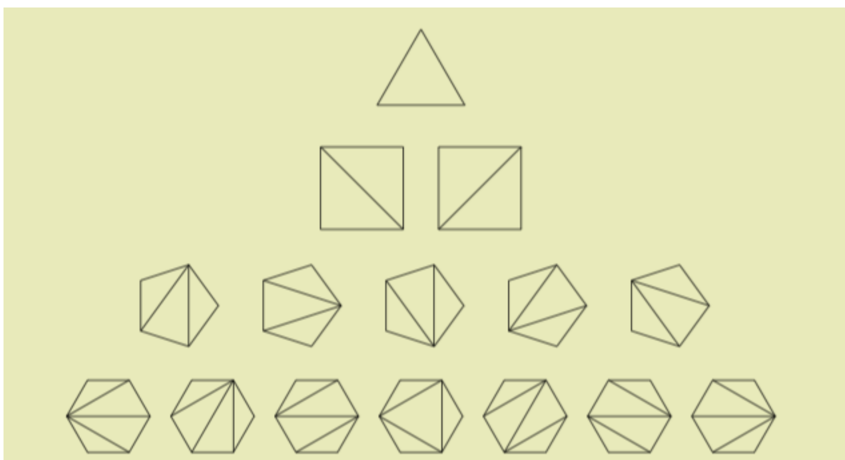
Lunch Problem (Triangulate!)
(Due after lunch at the door to the Math Bowl)

Name:________________ School:________________

A *triangulation* of a regular polygon is when you draw non-crossing line segments connecting corners of the polygon in such a way that you have cut the polygon into triangles. Let T_n be the number of triangulations of the regular polygon with n sides. For example, from the picture below we see that $T_3 = 1$, $T_4 = 2$, $T_5 = 5$. We have also shown you the first seven triangulations of the hexagon:



1. Please compute T_6, T_7, and T_8.
2. Please given a formula for T_n for any n. Note that your formula should depend on n. If nobody gives a correct formula, then the person who has computed the most T_n's correctly will be judged the winner.

The Answer: T_n is the (n-2)nd Catalan numbers. So

$$T_n = \frac{1}{n-1}\binom{2(n-2)}{n-2} = \frac{(2(n-2))!}{(n-1)!(n-2)!}.$$

In particular, $T_6 = 15$, $T_7 = 42$, and $T_8 = 132$