**Sooner Math Bowl 2024** **November 7, 2024**

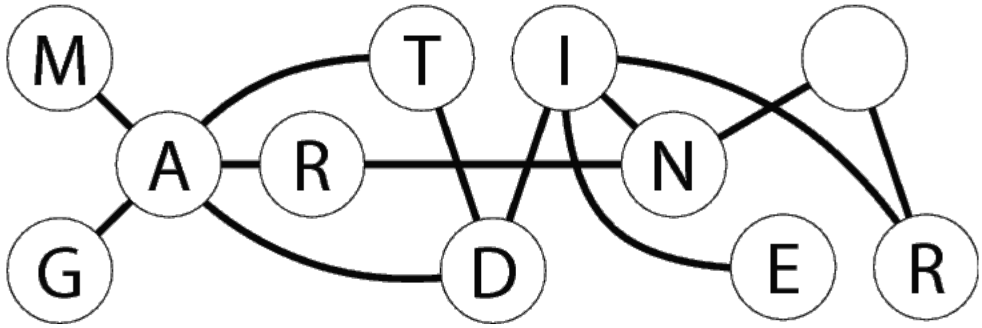



Photo Martin Gardner by Alex Bellos in 2008 in Norman

## Born in Tulsa in 1914 and passed away in Norman in 2010.

# Stage 1, Round 1 (2 Questions, 3 Minutes)

1. Two points in the plane have integer coordinates and the distance between them is $\sqrt{50}$. If these points form the opposite corners of a rectangle whose sides are parallel to the coordinate axes, what is the maximal area of such a rectangle?

   The Answer: 25

   Solution. Without loss of generality, one of the points can be chosen to be the origin $(0, 0)$ and the other point can be taken in the first quadrant with coordinates $(a, b)$ where $a^2 + b^2 = 50$. The number 50 can be written as the sum of two squares in two non-equivalent ways: $1^2 + 7^2$ and $5^2 + 5^2$. The area, $5\times 5 = 25$, will be maximal in the latter case.

2. If the integers $a, b, c$ and $d$ are such that

$$2^a 3^b 5^c 7^d - 76 = 2024,$$

   what is $a + b + c + d$?

   The Answer: 6

   Solution. The number $2024+76 = 2100$ can be factored as $3^1 7^1 2^2 5^2$, hence $a+b+c+d = 6$.

# Stage 1, Round 2 (Blitz Round, 3 Minutes)

a. If $a$ and $b$ are real numbers whose average is 10, what is the average of $a, b$, and 16?

The Answer: 12

Solution. Since $\frac{a+b}{2} = 10$, $a+b = 20$, and so $a+b+16 = 36$. Dividing by 3 shows that the average is 12.

b. What is the largest positive integer $k$ for which $3^k$ divides $\underbrace{999\ldots9}_{2024}$?

The Answer: $k = 2$

Solution. Let $N = \underbrace{999\ldots9}_{2024}$. Dividing by $3^2 = 9$, we have $N/9 = \underbrace{111\ldots1}_{2024}$. Notice, by the familiar rules for division by 3, $N/9$ is divisible by 3 if and only if the sum of its digits is so. The sum of these digits is 2024. By the same analysis, 2024 is not divisible by 3 since 2+0+2+4=8 is not divisible by 3. Thus, we can divide $N$ by $3^2$ but not $3^3$.

c. Which is more likely: rolling two fours with a fair die or getting 4 heads by tossing a fair coin?

The Answer: The chance of two fours is 1/36 while the four heads is 1/16. The four heads is more likely.

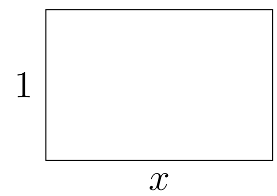
d. What is the first prime larger than 49?

The Answer: 53

e. If $\tan(\theta) = 5/9$, then what is $\cos(\theta)$?

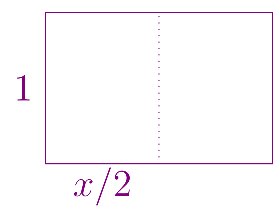
The Answer: If you compute using a right triangle you get that $\cos(\theta) = \frac{9}{\sqrt{106}}$.

f. Find a value $x > 1$ so that a 1 by $x$ rectangle can be cut into two congruent rectangles each similar to the original 1 by $x$ rectangle.




The Answer: $x = \sqrt{2}$

Solution. Cut the rectangle in half vertically:




The similarity condition then requires $\dfrac{x/2}{1} = \dfrac{1}{x}$, so $x^2 = 2$.

g. What is the sum of the first five primes?

The Answer: $2 + 3 + 5 + 7 + 11 = 28$.

# Stage 1, Round 3 (3 Questions, 5 Minutes)

1. If you reflect a line of slope $\pi$ over the y-axis, what is the slope of the reflected line?

   The Answer: $-\pi$

   Solution. Suppose $(x_0, y_0)$ and $(x_1, y_1)$ are on the original line. Then $(-x_0, y_0)$ and $(-x_1, y_1)$ are on the reflected line. The slope of the reflected line is then

$$\frac{y_1 - y_0}{(-x_1) - (-x_0)} = -\frac{y_1 - y_0}{x_1 - x_0} = -\pi.$$

2. If $x$ is twice the square of half of the square root of 2024, what is $x + 1$ (simplified)?

   The Answer: 1013

   Solution. Notice that $x = 2\left(\frac{1}{2}\sqrt{2024}\right)^2 = 2\left(\frac{1}{4} \cdot 2024\right) = \frac{1}{2} \cdot 2024 = 1012.$

3. Red stones weigh 1 pound while black stones weigh 2 pounds. How many different combinations of red and black stones weigh a total of 100 pounds? A combination must contain a minimum of one red stone and one black stone.

   The Answer: 49

   Solution. We only need consider how many black stones we use, then fill out the rest using red stones to get to 100 pounds. Since there must be at least one red stone, there can be anywhere from 1 to 49 black stones, yielding a total of 49 combinations.

# Stage 2, Round 1 (Blitz Round, 3 Minutes)

a. Suppose you have 10 distinct numbers. The average of the smallest 4 numbers is 20, while the average of the largest 6 numbers is 30. What is the average of all 10 numbers?

The Answer: 26

Solution. Since the average of the smallest 4 numbers is 20, their sum is 80. Similarly, the sum of the largest 6 numbers is 180. So the sum of all ten numbers is 260, and their average is 26.

b. If you have a square with side length $\ell$ whose perimeter equals its area, what is $\ell$?

The Answer: $\ell = 4$.

c. What is $x = 4^{2^{0^2}}$?

The Answer: $x = 4^{2^0} = 4^1 = 4$.

d. Consider the sequence $a_1 = -1, a_2 = 3, a_3 = 4, a_4 = 8, a_5 = 9, a_6 = 13, a_7 = 14, \ldots$.

If you continue this sequence, what is $a_{11}$?

The Answer: The change between terms alternates between 4 and 1. So every other term goes up by 5. Thus, we have $a_9 = 19$ and $a_{11} = 24$.

e. In a regular deck of 52 playing cards, is it more likely to draw a non-face card or a black card (i.e. club or spade)?

The Answer: non-face card

Solution. In the deck, there are 52/4=13 of each suit and so 26 black cards. The number of non-face cards is 10 is 4x10 = 40.

f. Write the number seventeen in base 2.

The Answer: $17 = 1 + 16 = 2^0 + 2^4$. In base 2, our number becomes 10001.

g. Determine the last digit of

$$M = 1! + 2! + 3! + \quad + 2024!$$

The Answer: 3.

Solution. Computing the first few terms we get $1! + 2! + 3! + 4! = 33$. Now $5! = 120$ ends with 0 and so do all subsequent terms as they are multiples of 5!. The last digit of $M$ is thus 3.

# Stage 2, Round 2 (3 Questions, 5 Minutes)

1. The positive integers $a$ and $b$ each have exactly two prime factors: 2 and 3. If $a$ does not divide $b$ and $b$ does not divide $a$, what is the smallest that $a$ can be?

   The Answer: 12

   Solution. Both $a$ and $b$ are of the form $2^n 3^m$ where $n$ and $m$ are at least 1. Thus 6 divides $a$. If $a = 6$ then $a$ will divide $b$. The next smallest option for $a$ is $2 \cdot 6 = 12$, and the pair $a, b = 12, 18$ satisfies the conditions.

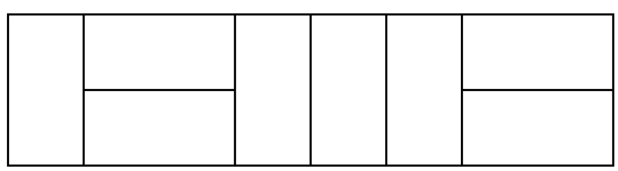
2. Carlos is looking at Rachel. Rachel is looking at Bob. Carlos is a Sooner. Bob is a Cowboy. (And, of course, Rachel is either a Cowboy or a Sooner, and no one is both!)

   True or False: A Sooner is looking at a person who is a Cowboy.

   The Answer: True.

   Solution. If Rachel is not a Sooner, then Carlos (a Sooner) is looking at Rachel (a Cowboy). If Rachel is a Sooner, then she (a Sooner) is looking at Bob (a Cowboy). Either way a Sooner is looking at a Cowboy.

3. A $2 \times N$ room is tiled with small rectangles of size $1 \times 2$. An example of tiling is given in the picture below. If $N = 8$, in how many ways can you tile the room?



   The Answer: 34

   Solution. Call $T_N$ the number of ways of tiling a $2 \times$ N room. Putting down our first times on the left, our tiling must start like one of the two examples.

or

One can observe that the sequence $T_N$ obeys a Fibonacci like recurrence: $T_{N+2} = T_{N+1} + T_N$. Finally, noting that $T_1 = 1$ while $T_2 = 2$, we iterate to see that the number of tilings are:

$$T_1 = 1,\ T_2 = 2,\ T_3 = 3,\ T_4 = 5,\ T_5 = 8,\ T_6 = 13,\ T_7 = 21,\ T_8 = 34.$$

# Stage 3, Round 1 (3 Questions, 5 Minutes)

1. The polynomnial

$$p(x) = x^5 - 4x^4 - 12x^3 + 34x^2 + 11x - 30$$

has roots $x = 1, -1, 2, 5$. Since $p(x)$ is degree 5 it must have one more root. What is it?

The Answer: $r = -3$

Solution. Call $r$ the missing root. We then have that $p(x) = (x-1)(x+1)(x-2)(x-5)(x-r)$. Plugging in zero we obtain $-30 = p(0) = -1 \cdot 1 \cdot 2 \cdot 5 \cdot r$. Solving gives $r = -3$.

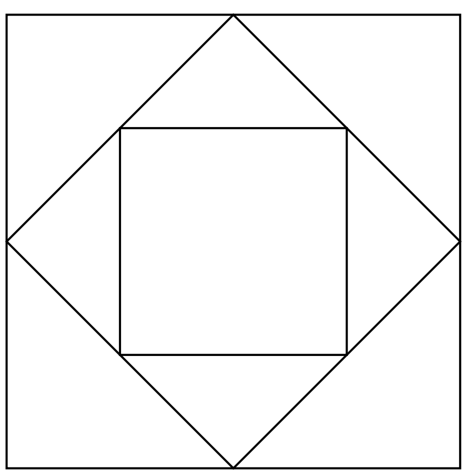
2. Find the integer $a$ that satisfies

$$a!(a+1)! = 10!$$

The Answer: 6

Solution. Note that the prime decomposition of 10! contains a unique copy of 7. We must then have a = 6, which does work.

3. In the figure below, the corners of each square touch the midpoint of the sides of the next larger square. If the center square has area 1 square inch, what is the area of the largest square?



The Answer: 4 square inches

Solution. The inner square has side length 1. Using the Pythagorean theorem twice we see that the outer square has side length 2, hence has area 4 square inches.

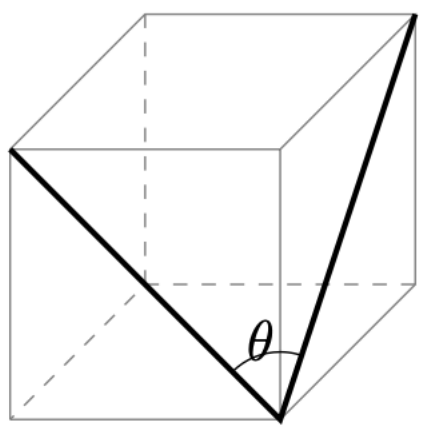
# Stage 3, Round 2 (3 Questions, 5 Minutes)

1. Compute the number $1 + 3 + 5 + \cdots + 17 + 19 + 21$.

   The Answer: 121

   Solution: The above is $(1+2+3+\cdots+21)-2(1+2+\cdots+10) = \frac{21 \cdot 22}{2} - 2 \cdot \frac{10 \cdot 11}{2} = 21 \cdot 11 - 10 \cdot 11 = 11 \cdot 11 = 121$.

2. What is the angle, in degrees, between two face diagonals of a cube meeting at the same corner?




   The Answer: 60°.

   Solution. Notice that adding a diagonal across the top face of the cube creates an equilateral triangle, so the angle must be 60°.

3. How many triangles have sides of length $\pi, \sqrt{2}$, and $a$, where $a$ is an integer? (The order of the sides is irrelevant.)

The Answer: 3

Solution. First note that $\pi \simeq 3.1$ and $\sqrt{2} \simeq 1.4$, so we have

$$\pi + \sqrt{2} \simeq 4.5 \quad \text{and} \quad \pi - \sqrt{2} \simeq 1.7.$$

By the triangle inequality, the other side $a$ must lie in the interval

$$1.6 < a < 4.6.$$

Notice that we expanded our interval a little to account for any error in our approximations of $\pi$ and $\sqrt{2}$. So, we see that $a$ must be one of $2, 3$, or $4$. That is, there are 3 triangles.

# The End!

## Basic Instructions:

- Each team can take one or two packets for each round. This should make it easier for everyone to see the questions and have extra scratch paper.
- However, you will turn in ONLY ONE answer sheet per team per round.

**Sooner Math Bowl 2024** **November 7, 2024**

# Spot Prize II (Word Search!)

## Name:__________________ School:__________________

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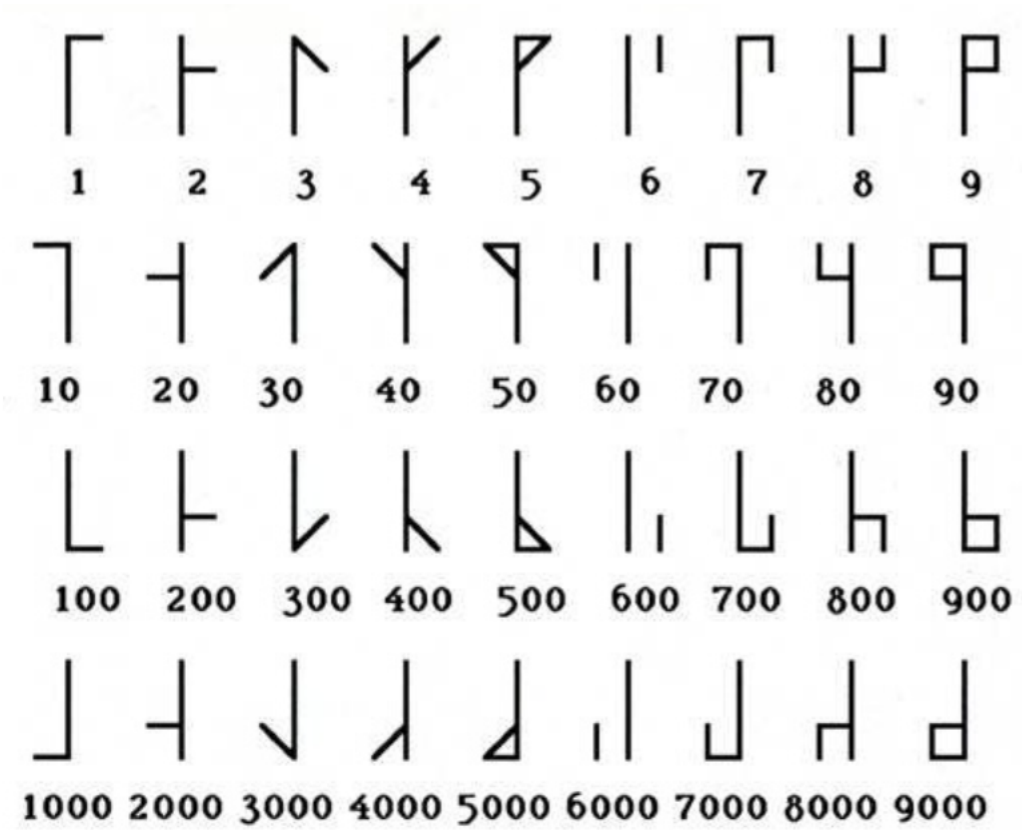
| G | Y | N | L | A | I | M | O | N | Y | L | O | P | H | L | P | E | O | H | K | R | X |
| D | A | O | D | O | O | O | Z | U | H | I | B | E | L | I | H | Z | K | U | W | E | W |
| R | N | R | B | R | E | T | A | T | S | K | O | O | N | T | L | T | I | F | J | D | Y |
| L | A | A | D | W | A | T | H | G | J | R | R | N | E | J | O | W | U | F | A | E | W |
| D | A | V | L | N | O | S | P | T | E | R | I | O | Q | A | R | E | F | L | K | O | D |
| K | G | O | Q | O | E | C | D | M | A | N | N | C | N | V | I | F | J | E | S | R | C |
| V | C | O | E | Z | R | R | X | C | G | P | D | A | M | L | N | W | Q | G | N | A | Y |
| N | O | I | T | A | T | U | M | R | E | P | I | F | C | E | C | C | P | U | C | Z | M |
| T | A | H | C | A | V | U | J | Z | R | D | A | W | Y | B | Z | Q | I | B | J | Y | M |
| N | L | Q | C | U | N | R | C | Q | N | V | R | O | W | A | V | U | U | R | Z | T | E |
| O | G | X | K | I | C | U | H | I | V | C | K | O | C | Z | E | E | E | Y | C | I | A |
| R | E | I | B | L | G | U | Z | C | F | L | D | E | S | L | E | E | H | W | N | L | L |
| M | B | E | P | U | B | A | A | M | A | O | C | C | R | F | V | T | E | F | Y | I | E |
| A | R | S | N | P | P | R | M | H | N | V | G | U | J | O | A | Q | I | C | M | B | X |
| N | A | T | I | M | D | B | O | P | S | A | H | V | I | M | T | Z | C | N | M | A | A |
| Y | P | Q | K | S | V | M | E | B | I | P | U | Z | Z | L | E | N | X | Z | O | B | K |
| Y | K | Q | O | Y | A | T | X | R | W | P | H | E | N | V | D | V | A | K | K | O | V |
| A | Z | L | L | K | R | B | T | C | E | S | P | D | V | H | D | I | R | C | V | R | F |
| D | E | N | A | O | L | E | B | V | L | S | W | N | O | Y | F | F | C | E | S | P | X |
| Q | L | I | V | L | W | G | C | R | X | Z | X | M | A | R | T | I | N | Y | V | Y | U |
| F | G | X | Q | P | O | X | B | W | S | G | S | Y | S | P | H | E | R | I | C | A | L |

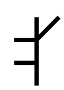
- NIKOLA
- PETROV
- OKSTATE
- COWBOYS
- LEWIS
- CARROLL
- SPHERICAL
- ANDRAS
- LORINCZ
- SPINNING
- WHEELS
- CANTOR
- EMMY
- NOETHER
- MARTIN
- GARDNER
- SHUFFLE
- CARDS
- PERMUTATION
- MAGIC
- PROBABILITY
- MATH
- PUZZLE
- TULSA
- NORMAN
- OKLAHOMA
- POLYNOMIAL
- ROLAND
- ROEDER
- INDIANA
- SQUARE
- CIRCLE
- ALGEBRA
- THEOREM

# Spot Prize I (Break the Code!)

**Name:**__________________ **School:**__________________

During the Middle Ages, Cistercian monks developed an interesting additive numeration system where each number from 1 to 9999 could be expressed as a single symbol. Their convention is illustrated in the table below:




For example, 2024 would be reprsented by  and 1001 by . If X is the largest multiple of 4 whose Cistercian notation is invariant under a 180° rotation, what is X?

The Answer: 8888

Solution. Rotating a cistercian symbol by 180° permutes the units with the thousands and the tens with the hundreds. Such a number is thus invariant under rotation if and only if it is of the form abba where a and b represent digits. A number is divisible by 4 if and only if its last two digits are divisible by 4. Given the palindromic nature of the number we are looking for, we seek for the multiple of four with the largest possible unit digit and then with the largest possible ten digit. This number is clearly 88 as multiples of 4 are even, hence X = 8888.

**OU Math Day 2024** **November 7, 2024**

# Lunch Problem

## Name:________________ School:________________

## Due after lunch at the door to the Math Bowl.

## Write your solution on the back.

Amongst four friends, Alice, Bob, Charly and Donna, each person either always lies or always tells the truth. One evening, they make the following statements:

Alice - Bob is a liar!

Charly - Alice is a liar.

Donna - Alice and Charly are both liars. Bob is a liar!

Who are the liars? Pick from the options below, then do your best to explain your reasoning.

(A) Alice & Bob (B) Alice & Charlie (C) Alice & Donna

(D) Bob & Donna (E) Charlie & Donna

The Answer: Alice and Donna

Solution. Assume Donna says the truth. Then, Alice is a liar (according to Donna's statement) so Bob must tell the truth (since Alice is liar). However, this contradicts Donna's statement. Therefore, our assumption is wrong and Donna is a liar. Hence, Bob tells the truth (since Donna is liar) and so Alice is also liar. Lastly, Charlie tells the truth as Alice is liar. As a consequence, Alice and Donna are the two liars.